Zero-fermion modes in models with spontaneous symmetry-breaking

1979 J. Phys. A: Math. Gen. 12 L343

(http://iopscience.iop.org/0305-4470/12/12/005)

J. Phys. A: Math. Gen., Vol. 12, No. 12, 1979. Printed in Great Britain

LETTER TO THE EDITOR

Zero-fermion modes in models with spontaneous symmetry-breaking

N V Krasnikov, V A Rubakov and V F Tokarev
Institute for Nuclear Research, Academy of Science, Moscow, USSR

Received 23 July 1979

Abstract. It is shown that zero-fermion modes exist in the external field with non-zero topological charge in Weinberg–Salam type models. An explicit expression for the zero modes is found in a particular case. Some physical consequences of the existence of the zero modes are discussed.

The topological configurations of gauge-field instantons (Belavin *et al* 1975) have been intensively investigated during the last few years. The essential property of these configurations is the existence of definite-chirality zero-fermion modes in the external field with non-zero topological charge ('tHooft 1976a,b, Brown *et al* 1977, Jackiw and Rebbi 1977, Nielsen and Schroer 1977, Schwarz 1977). The existence of the zero modes leads to the dynamical $v(1)$ breaking ('tHooft 1976a, b), and to fermion number non-conservation in V-A models (Peccei and Quinn 1977, Krasnikov *et al* 1978, 1979).

The investigation of the topological configurations in models with spontaneous symmetry breaking is of particular interest (Callan *et al* 1977, Inoue 1978, Cho 1979). In this letter we consider the zero-fermion modes in the Weinberg–Salam type model. We show that the zero modes do exist and present their explicit expression in a particular case.

Consider the model with the Lagrangian

$$\mathscr{L} = \mathscr{L}A, \phi + \mathscr{L}F \tag{1}$$

in Euclidean space–time. $\mathscr{L}A, \phi$ is the standard Lagrangian of the SU(2) gauge field $A_\mu = (e/2i)A^a_\mu \tau^a$ and the Higgs doublet $\phi, \mathscr{L}F$ is the Lagrangian of spinor fields.

$$\mathscr{L}F = \psi^+\gamma_\mu(\partial_\mu + A_\mu\pi^+)\psi + u^+\gamma_\mu\partial_\mu u + v^+\gamma_\mu\partial_\mu v$$
$$-[\mu(\psi^+\phi)\pi^- u + \tilde{\mu}(\psi^+\tilde{\phi})\pi^- v + \text{HC}]. \tag{2}$$

In (2) ψ is an SU(2) doublet, u and v are SU(2) singlets and $\tilde{\phi} = i\tau_2\phi^*$. The Euclidean γ matrices are

$$\gamma_\mu = \begin{pmatrix} 0 & s_\mu \\ s^+_\mu & 0 \end{pmatrix}, \qquad s_\mu = (1, i\sigma_i);$$

σ_i are Pauli matrices acting on spatial variables. The matrices $\pi^\pm$ are defined by the expressions

$$\pi^\pm = \frac{1 \pm \gamma_5}{2}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Equations for the zero modes are

$$\gamma_\mu(\partial_\mu + A_\mu \pi^+)\psi = \mu\phi\pi \; u + \tilde{\mu}\tilde{\phi}\pi^- v,$$
$$\gamma_\mu \partial_\mu u = \mu\pi^+(\phi^+\psi), \qquad \gamma_\mu\partial_\mu v = \tilde{\mu}\pi^+(\tilde{\phi}\psi), \tag{3}$$

where the fields A_μ, ϕ are the external ones. In the model (1) the divergence of the fermionic current $\mathscr{J}_F^\mu$ is anomalous:

$$\partial_\mu \mathscr{J}_F^\mu = (32\pi^2)^{-1}\,\mathrm{Tr}\,\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}.$$

The change of the fermion number in the external field is connected with the topological charge† q:

$$\Delta N_F = q, \qquad q = \frac{1}{32\pi^2}\,\mathrm{Tr}\int \mathrm{d}^4x\,\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}. \tag{4}$$

Equation (4) is a signal of the existence of normalisable zero-fermion modes in the external field with $q \neq 0$ (cf 'tHooft 1976b, Peccei and Quinn 1977).

To see the existence of the zero modes in a more rigorous way we proceed as follows. We rewrite equations (3) in the compact form

$$\mathbf{D}\mathbf{P}^+\boldsymbol{\Phi} = \mathbf{0}$$

where Φ is constructed from ψ, u, v:

$$\Phi = \begin{pmatrix}\psi \\ u \\ v\end{pmatrix}.$$

The operators $\mathbf{D}$ and $\mathbf{P}^\pm$ are the matrices

$$\mathbf{D} = \begin{pmatrix}\gamma_\mu(\partial_\mu + A_\mu) & -\mu\phi & -\tilde{\mu}\tilde{\phi} \\ -\mu\phi^+ & \gamma_\mu\partial_\mu & 0 \\ -\tilde{\mu}\tilde{\phi}^+ & 0 & \gamma_\mu\partial_\mu\end{pmatrix}, \qquad \mathbf{P}^\pm = \begin{pmatrix}\pi^\pm & 0 & 0 \\ 0 & \pi^\mp & 0 \\ 0 & 0 & \pi^\mp\end{pmatrix}.$$

Consider the function

$$f(m) = \mathrm{Tr}\left[\frac{m}{-\mathbf{D}^+\mathbf{D} + m}(\mathbf{P}^+ - \mathbf{P}^-)\right]$$

where the trace is to be taken in the functional space. By analogy with Brown *et al* (1977) one finds the following properties of $f(m)$:

$$\mathrm{d}f/\mathrm{d}m = 0, \qquad f(0) = n_+ - n_-, \qquad f(\infty) = q,$$

where $n_\pm$ denotes the number of zero modes of the operator $\mathbf{D}$. Thus $n_+ - n_- = q$ is in perfect agreement with equation (4).

We choose the normalisation of the Higgs field so that $|\langle\phi\rangle_{\mathrm{vac}}| = 1$. Then μ and $\tilde{\mu}$ become the masses of the u and v quarks respectively. We shall now describe an explicit form of the solution of equations (3) in a particular case $\mu = \tilde{\mu}$ and spherically symmetric configuration of the external fields ('tHooft 1976a)

$$A_\mu = \frac{-\mathrm{i}\eta_{\mu\nu a}x_\nu\tau_a}{\rho^2 + x^2}, \qquad \phi = \frac{\tau_\nu x_\nu}{(\rho^2 + x^2)^{1/2}}\phi_0, \tag{5}$$

† The topological charge of the Higgs field ϕ is equal to the topological charge of the gauge field for finite-action configurations (Romanov and Fateev 1977, Woo 1977a,b).

where $\eta_{\mu\nu a}$ are 'tHooft symbols, $\tau_\nu = (1, i\tau_i)$, ϕ_0 is a constant SU(2) spinor, $\phi_0^+\phi_0 = 1$ and ρ is the instanton size.

Equations (3) in this case become

$$s_\mu^+ \partial_\mu \psi' - \frac{i s_\mu^+ \eta_{\mu\nu a} \tau_a x_\nu}{\rho^2 + x^2} \psi' = \mu \left(\frac{\tau_\nu x_\nu}{(\rho^2 + x^2)^{1/2}} \phi_0 u' + \frac{\tau_\nu x_\nu}{(\rho^2 + x^2)^{1/2}} \tilde{\phi}_0 v' \right), \tag{6}$$

$$s_\mu \partial_\mu u' = \mu \left(\phi_0^+ \frac{\tau_\nu^+ x_\nu}{(\rho^2 + x^2)^{1/2}} \psi' \right), \qquad s_\mu \partial_\mu v' = \mu \left(\tilde{\phi}_0^+ \frac{\tau_\nu^+ x_\nu}{(\rho^2 + x^2)^{1/2}} \psi' \right).$$

In equations (6) the two-component spinors ψ, u, v, were used:

$$\psi = \begin{pmatrix} \psi' \\ 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 \\ u' \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ v' \end{pmatrix}.$$

We choose the *ansatz*

$$\psi'_{\alpha i} = f(r^2)(\tau_2)_{\alpha i}, \qquad u_\alpha = q(r^2)(\tau_2)_{\alpha i} \phi_{0i}, \qquad v_\alpha = q(r^2)\phi_{0\alpha}. \tag{7}$$

On the left-hand side of equations (7) α and i are spatial and gauge group indices respectively. With this *ansatz* we obtain

$$2\mathrm{d}q/\mathrm{d}r^2 = \mu f(\rho^2 + r^2)^{1/2}, \qquad \mu(\rho^2 + r^2)^{1/2} q = 2\mathrm{d}f/\mathrm{d}r^2 + 3f/(\rho^2 + r^2). \tag{8}$$

Equations (8) indeed have a normalisable solution, namely

$$\begin{aligned} f &= [\rho\mu/(\rho^2 + r^2)^{1/2}] C(\rho\mu) \mathrm{K}_2[\mu(\rho^2 + r^2)^{1/2}], \\ q &= -\rho\mu C(\rho\mu) \mathrm{K}_1[\mu(\rho^2 + r^2)^{1/2}]/(\rho^2 + r^2)^{1/2}, \end{aligned} \tag{9}$$

where $\mathrm{K}_{1,2}$ are Bessel functions and C is a normalisation constant. In the physically important case of 'tHooft (1976a) $\rho\mu \to 0$, C is $\rho\mu$-independent.

In the limit $\rho\mu \to 0$, $\tau\mu \to 0$ (no interaction with Higgs field) this solution reproduces that of 'tHooft (1976a):

$$f \to \rho/(\rho^2 + r^2)^{3/2}, \qquad q \to 0.$$

The functions f, q vanish exponentially at large distances.

In the limit $\rho\mu \to 0$ a Fourier transform of the solution (9) has the form

$$f(k) = C'(\rho/\mu^2) F(\tfrac{5}{2}, \tfrac{1}{2}, 2, -k^2/\mu^2) \tag{10}$$

where C' is an unimportant numerical constant and F a hypergeometric function. Note that $f(k)$ has a pole at $k^2 = -\mu^2$ with a residue proportional to ρ.

The existence of the zero-fermion modes in Weinberg–Salam type models means that the sectors with a non-zero topological charge q contribute only to the Green functions with fermion number non-conservation $\Delta N_{\mathrm{F}} = q n_{\mathrm{f}}$ in the case of n_{f} fermionic doublets. ρ dependence of the described type (see equation (10)) leads to the fact that these Green functions are exponentially small. The integral over instanton size ρ is convergent in the upper limit ('tHooft 1976a) due to the Higgs field factor $\exp(-\rho^2 m_{\mathrm{n}}^2)$, while in the lower limit it converges due to the factor ρ in the Fourier transform (10). Thus, for instance, the contribution of the $q = 1$ sector is proportional to

$$\exp(-8\pi^2/g_{\mathrm{w}}^2)$$

where $g_{\mathrm{w}} = \bar{g}(p^2 = m_{\mathrm{w}}^2)$ is an effective coupling. In conclusion we would like to stress the following point. It has been shown by Callan *et al* (1977), Polyakov (1977) and

Inoue (1978) in the dilute gas approximation that the global symmetry in the two-dimensional Higgs model and the three-dimensional Georgy–Glashow model is restored due to instantons, while the gauge symmetry remains broken and the vector field remains massive. In the realistic four-dimensional models the analysis of instanton effects has not yet been performed, mainly because of the nonexistence of classical solutions with finite action. However, the topological configurations with finite action do exist, so global symmetry restoration due to instantons in four dimensions can also be expected (see also Cho 1979). On the other hand every realistic unified model of weak and electromagnetic interactions contains initially massless fermion fields, which become massive after spontaneous symmetry breaking. As we have shown in the case of the Weinberg–Salam model, these fermion fields have zero modes in the presence of instantons, thus leading to a vanishing instanton contribution to fermion–number conserving Green's functions, in particular, to the effective potential. Thus symmetry restoration due to instantons cannot take place in the models with fermions.

We are indebted to V A Matveev and A N Tavkhelidze for useful discussions.

References

Belavin A A, Polyakov A M, Schwarz A S and Tyupkin Yu S 1975 *Phys. Lett.* B **59** 85
Brown L S, Carlitz R D and Lee C 1977 *Phys. Rev.* **D16** 417
Callan C G, Dashen R and Gross D 1977 *Phys. Rev.* **D16** 2526
Cho Y M 1979 *Phys. Lett.* B **82** 25
'tHooft G 1976a *Phys. Rev.* **D14** 3432
—— 1976b *Phys. Rev. Lett.* **37** 8
Inoue T 1978 *Prog. Theor. Phys.* **59** 986
Jackiw R and Rebbi C 1977 *Phys. Rev.* **D16** 1052
Krasnikov N V, Rubakov V A and Tokarev V F 1978 *Phys. Lett.* B **79** 423
—— 1979 *Yad. Fiz.* **29** 1127
Nielsen N K and Schroer B 1977 *Nucl. Phys.* **B127** 493
Peccei R D and Quinn H 1977 *Nuovo Cim.* **41**A 309
Polyakov A M 1977 *Nucl. Phys.* **B120** 423
Romanov N V and Fateev V A 1977 *Teor. Mat. Fiz.* **33** 286
Schwarz A S 1977 *Phys. Lett.* B **67** 172
Woo G 1977a *J. Math. Phys.* **18** 1756
—— 1977b *Phys. Rev.* **D16** 1014